

EXPONENTIAL FUNCTION WITH ARBITRARY GRAININESS FUNCTION

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Abstract: In this paper we introduce the exponential function on arbitrary graininess function on Time Scales. In the introduction of the paper, we provide the basic theory of the Time Scale, where Time Scales calculus unifies the continuous and discrete calculus and extends it to any nonempty closed subset of the real numbers. The exponential function is an integral part of the Laplace transform, and for this reason its study is very important. We base our paper on its extension to the Time Scale calculus, because the Laplace transform has recently been studied on a generalized Time Scale. We are interested in the asymptotic behavior of these functions. We also present various properties of the exponential function and offer some examples.

Key words: Generalized Time Scale, Graininess function, Exponential function, Delta derivative, Regressive function

1. INTRODUCTION

A Time Scale, which inherits the standard topology on $\mathbb{R}$, is a nonempty closed subset of reals. Time Scales calculus unifies the continuous and discrete calculus and was introduced by Stefan Hilger in 1988 [1]. extends it to any nonempty closed subset of the real numbers. Throughout this paper, will be denoted by the symbol $\mathbb{T}$, and the intervals with a subscript $\mathbb{T}$ are used to denote the intersection of the usual interval with $\mathbb{T}$. In this subsection, the basic definitions of Time Scales calculus are introduced based on the introductory book [3]. It includes both the classical derivative and the forward difference operator as special cases. It also includes Riemann integrals and finite sums as inverse operations.

Definition 1 ([2]) For $t \in T$ we define the forward jump operator $\sigma: T \rightarrow T$ as $\sigma(t) = \inf\{s \in T \mid s > t\}$.

We note that $\sigma(t) \geq t$ for all $t \in T$.

Definition 2 ([2]) If $f: T \rightarrow \mathbb{R}$ is a function, then we define the function $f^\sigma(t): T \rightarrow \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$.

Definition 3 ([3]) The mapping $\mu: T \rightarrow [0, \infty)$, such that $\mu(t) := \sigma(t) - t$ is called the graininess.

Definition 4 ([4]) The minimal graininess function $\mu^*: \mathbb{T} \rightarrow \mathbb{R}$ is defined by

$$\mu^*(s) = \inf_{\tau \in [s, \infty) \cap T} \mu(\tau).$$

Definition 5 ([3]) Assume $f: T \rightarrow \mathbb{R}$ is a function and let $t \in T$. Then we define $f^\Delta(t)$ to be Δ the number with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the Δ (or Hilger) derivative of f at t .

Theorem 1 ([3]) Assume f is a function and let $t \in T$. Then we have following:

1. If f is differentiable at t , then f is continuous at t ;
2. If f is continuous at t and t is right-scattered, then f is differentiable at t with $f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$.

Theorem 2 ([2]) Assume $f, g: T \rightarrow \mathbb{R}$ are Δ -differentiable at $t \in T$. Then the following hold:

1. $(\alpha f(t) + \beta g(t))^\Delta = \alpha f^\Delta(t) + \beta g^\Delta(t)$;

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$$2. (f(t)g(t))^{\Delta} = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t).$$

Definition 6 ([2]) A function $f: T \rightarrow R$, is called regulated provided its right-sided limits exist (finite) at all right-dense points in T and its left-sided limits exist (finite) at all left-dense points in T .

Definition 7 ([2]) A function $f: T \rightarrow R$, is called rd -continuous provided it is continuous at right-dense points in T and its left-sided limits exist (finite) at left-dense points in T . The set of rd -continuous functions $f: T \rightarrow R$, will be denoted by $C_{rd} = C_{rd}(T)$.

Definition 8 ([5]) We define the Cauchy integral by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \quad \text{for all } a, b \in T.$$

A function $F: T \rightarrow R$ is called an antiderivative of $f: T \rightarrow R$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in T$.

Theorem 3 ([5]) If $a, b, c \in T$, $\alpha \in R$ and $f, g \in C_{rd}$, then

1. $\int_a^b (\alpha f(t) + \beta g(t))\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t;$
2. $\int_a^b f(t)\Delta t = - \int_b^a f(t)\Delta t;$
3. $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t;$
4. $\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t;$
5. $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t;$
6. $\int_a^a f(t)\Delta t = 0.$

2. EXPONENTIAL FUNCTION ON TIME SCALE

Definition 9 ([2]) For $h > 0$ we define the Hilger complex numbers as

$$C_h = \left\{ z \in \mathbb{C} \mid z \neq -\frac{1}{h} \right\}.$$

Theorem 4 ([2]) If we define the "circle plus" addition $\oplus$ on C_h by

$$z \oplus \omega := z + \omega + z\omega h$$

then $(C_h, \oplus)$ is an Abelian group.

Theorem 5 ([2]) If $z \in C_h$, then the additive inverse of z under the operation $\oplus$ is

$$\ominus z = \frac{-z}{1+zh}.$$

Definition 10 ([2]) We say that a function $p: T \rightarrow R$ is regressive provided

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all } t \in T$$

holds. The set of all regressive and rd -continuous function $f: T \rightarrow R$ will be denoted by

$$\mathfrak{R} = \mathfrak{R}(T) = \mathfrak{R}(R, T).$$

If $1 + \mu(t)p(t) > 0$ for all t , then we say that p is positively regressive and that $p \in \mathfrak{R}^+(R, T)$. We use the notation $\mathfrak{R}_c$ to denote constant regressive functions.

Lemma 1 ([2]) If $p, q \in \mathfrak{R}$, then the function $p \oplus q$ and the function $\ominus p$ defined by

$$(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t), \tag{1}$$

$$(\ominus p)(t) = \frac{-p(t)}{1+\mu(t)p(t)}, \quad \text{for all } t \in T \quad p, q \in \mathfrak{R}. \quad (2)$$

Definition 11 ([2]) For $h > 0$, we define the cylinder transformation $\xi_h: C_h \rightarrow Z_h$ by

$$\xi_h(z) = \frac{1}{h} \text{Log}(1 + hz)$$

where Log is the principal logarithm function.

Definition 12 ([3]) If $p \in \mathfrak{R}$, then we define the exponential function by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta\tau\right) \quad \text{for all } s, t \in T.$$

Lemma 2 ([3]) If $p \in \mathfrak{R}$ and $r, s, t \in T$ then:

1. $e_p(t, t) = 1$, and $e_0(t, s) = 1$;
2. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
3. $e_p(t, r)e_p(r, s) = e_p(t, s)$;
4. $e_p(t, s) = \frac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$;
5. $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$;
6. $\frac{e_p(t, s)}{e_q(t, s)} = e_{p \ominus q}(t, s)$;
7. $e_p^\Delta(t, s) = p(t)e_p(t, s)$.

Theorem 6 ([4]) If $p, q \in \mathfrak{R}$, then

$$e_{p \ominus q}^\Delta(t, s) = (p - q) \frac{e_p(t, s)}{e_q^\sigma(t, s)}$$

Proof. We have

$$\begin{aligned} e_{p \ominus q}^\Delta(t, s) &= (p \ominus q)(t)e_{p \ominus q}(t, s) \\ &= \frac{p(t) - q(t)}{1 + \mu(t)q(t)} \frac{e_p(t, s)}{e_q(t, s)} = \frac{(p(t) - q(t))e_p(t, s)}{e_q(\sigma(t), s)} \end{aligned}$$

Theorem 7 ([4]) (Decay of Exponential Function) Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = \infty$ and $s \in \mathbb{T}$ and $p \in \mathfrak{R}_c^+([s, \infty), T)$. Then for any $z \in C_{\mu^*(s)}(p)$,

1. $|e_{p \ominus z}(t, s)| \leq e_{p \ominus \text{Re}_{\mu^*(s)}(z)}(t, s)$;
2. $\lim_{t \rightarrow \infty} e_{p \ominus \text{Re}_{\mu^*(s)}(z)}(t, s) = 0$;
3. $\lim_{t \rightarrow \infty} e_{p \ominus z}(t, s) = 0$.

2.1. Graininess function and Exponential function on arbitrary Time Scale

In this section we represent the exponential function for some specific Time Scales. Consider the Time Scale:

1. $\mathbb{T} = \mathbb{R}$:

The graininess function is: $\mu(t) = 0$;

Exponential function is: $e_p(t, s) = e^{p(t-s)}$.

2. $\mathbb{T} = \ln \mathbb{N}$:

The graininess function is: $\mu(t) = \ln(e^t + 1) - t$;

Exponential function is: $e_p(t, s) = C e^{pt} [1 + O(e^{-t})]$

3. $\mathbb{T} = \mathbb{N}_0^r, r \in (0, \frac{1}{2})$:

The graininess function is: $\mu(t) = t \left[\left(1 + t^{-\frac{1}{r}}\right)^r - 1 \right]$

Exponential function is: $e_p(t, s) = C e^{pt} \left[1 + O\left(\frac{1}{t^{\frac{1}{r}-2}}\right) \right]$

4., $\mathbb{T} = \sqrt{\mathbb{N}_0}$

The graininess function is: $\mu(t) = \sqrt{t^2 - 1} - t$

Exponential function is: $e_p(t, s) = C e^{pt} t^{-\frac{p^2}{4}} \left[1 + O\left(\frac{1}{t}\right) \right]$

5. $\mathbb{T} = \mathbb{N}_0^r, r \in (\frac{1}{2}, \frac{2}{3})$:

The graininess function is: $\mu(t) = t \left[\left(1 + t^{-\frac{1}{r}}\right)^r - 1 \right]$

Exponential function is: $e_p(t, s) = C e^{pt} e^{-\frac{p^2 r^2}{2(2r-1)} t^{(2r-1)/r}} \left[1 + O\left(\frac{1}{t^{\frac{2}{r}-3}}\right) \right]$

6. $\mathbb{T} = \mathbb{N}_0^{2/3}$:

The graininess function is: $\mu(t) = t \left[\left(1 + t^{-\frac{3}{2}}\right)^{\frac{2}{3}} - 1 \right]$;

Exponential function is: $e_p(t, s) = C e^{pt} e^{-\frac{2p^2}{3} \sqrt{t}} t^{\frac{4p^3}{27}} \left[1 + O\left(\frac{1}{\sqrt{t}}\right) \right]$.

7. $\mathbb{T} = \mathbb{N}_0$:

The graininess function is: $\mu(t) = 1$;

Exponential function is: $e_p(t, s) = (1 + p)^{t-s}$.

8. $\mathbb{T} = \mathbb{N}_0^r, r = 2, 3$:

The graininess function is: $\mu(t) = \left[\left(t^{\frac{1}{r}} + 1\right)^r - t \right]$;

Exponential function is: $e_p(t, s) = C (pr)^{\frac{1}{r}} e^{(r-1)t^{\frac{1}{r}} [\ln(t^{1/r}) - 1]} [1 + O(1)]$.

9. $\mathbb{T} = q^{\mathbb{N}_0}, q > 1$:

The graininess function is: $\mu(t) = (q - 1)t$;

Exponential function is: $e_p(t, s) = C (p(q - 1))^{\log_q t} t^{\frac{\log_q q^{t-1}}{2}} \left[1 + O\left(\frac{1}{t}\right) \right]$.

3. AN EXAMPLE OF EXPONENTIAL FUNCTION ON VARYING TIME SCALE

Example. Let Time Scale is: $\mathbb{T}_3 = [0, 0.45] \cup [0.55, 6]$ and $p = \frac{1}{2}$, $t_0 = 0$. Then:

1. For $t \in [0, 0.45]$ is $\sigma(t) = t$. We obtain:

$$\begin{aligned} e_{\frac{1}{2}}(t, 0) &= \exp\left(\int_0^t \xi_{\mu(\tau)}\left(\frac{1}{2}\right) \Delta\tau\right) = \exp\left(\int_0^t \left(\frac{1}{2}\right) \Delta\tau\right) \\ &= \exp\left(\int_0^t \left(\frac{1}{2}\right) \Delta\tau\right) = \exp\left(\frac{1}{2}t\right) \end{aligned}$$

2. For $t = 5.55$ is $\sigma(0.45) = 5.55$. Then:

$$\begin{aligned} e_{\frac{1}{2}}(5.55, 0) &= \exp\left(\int_0^{5.55} \xi_{\mu(\tau)}\left(\frac{1}{2}\right) \Delta\tau\right) \\ &= \exp\left(\int_0^{0.45} \xi_0\left(\frac{1}{2}\right) \Delta\tau + \int_{0.45}^{5.55} \xi_{5.10}\left(\frac{1}{2}\right) \Delta\tau\right) \\ &= \exp\left(\int_0^{0.45} \left(\frac{1}{2}\right) \Delta\tau + \int_{0.45}^{5.55} \frac{\text{Log}\left(1 + \frac{5.10}{2}\right)}{5.10} \Delta\tau\right) \\ &= \exp\left(\frac{0.45}{2} + \ln\left(1 + \frac{5.10}{2}\right)\right) = \left(1 + \frac{5.10}{2}\right) \exp\left(\frac{0.45}{2}\right) \end{aligned}$$

3. For $t \in (5.55, 6]$ is $\sigma(t) = t$:

$$\begin{aligned} e_{\frac{1}{2}}(t, 0) &= \exp\left(\int_0^t \xi_{\mu(\tau)}\left(\frac{1}{2}\right) \Delta\tau\right) \\ &= \exp\left(\int_0^{0.45} \left(\frac{1}{2}\right) \Delta\tau + \int_{0.45}^{5.55} \ln\left(1 + \frac{5.10}{2}\right) \Delta\tau + \int_{5.55}^t \left(\frac{1}{2}\right) \Delta\tau\right) = \\ &= \exp\left(\frac{0.45}{2} + \ln\left(1 + \frac{5.10}{2}\right) + \frac{1}{2}t - \frac{5.55}{2}\right) \\ &= \exp\left(\frac{0.45}{2}\right) \left(\ln\left(1 + \frac{5.10}{2}\right)\right) \cdot \exp\left(\frac{1}{2}t - \frac{5.55}{2}\right) \end{aligned}$$

4. For $t \in (5.55, 6]$ we obtain:

$$\begin{aligned} e_{\frac{1}{2}}(5.55, t) &= \exp\left(\int_{5.55}^t \xi_0\left(\frac{1}{2}\right) \Delta\tau\right) \\ &= \exp\left(\int_{5.55}^t \frac{1}{2} \Delta\tau\right) = \exp\left(\frac{t}{2} - \frac{5.55}{2}\right). \end{aligned}$$

4. CONCLUSION

The theory of Time Scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD thesis in 1988 in order to unify continuous and discrete analysis. The Time Scales calculus has a tremendous potential for applications. One of the applications of this calculus is to the Laplace transform used in signal theory. In this way, the Laplace transform and the z-transform are considered at once. Bohner and Peterson defined the Laplace transform on Time Scales, denote by $\mathcal{L}\{f(t)\}(z)$, in form $F(z) = \mathcal{L}\{f(t)\}(z) = \int_0^\infty f(t) e_{\ominus z}^\sigma(t, 0) \Delta t$. By taking $\mathbb{T} = \mathbb{R}$, from this form of the Laplace transform it is the classical Laplace transform $F(z) = \mathcal{L}\{f(t)\}(z) = \int_0^\infty e^{-zt} f(t) dt$,

while by taking $\mathbb{T} = \mathbb{Z}$, it is the $\mathcal{Z}$ -transform in the form $F(z) = \mathcal{L}\{f[n]\}(z) = \sum_{n=0}^{\infty} f[n]z^{-(n+1)} = \frac{\mathcal{Z}\{f[n]\}(z+1)}{z+1}$, where $\mathcal{Z}\{f[n]\}(z) = \sum_{n=0}^{\infty} f[n]z^{-n}$ is the classical form of the $\mathcal{Z}$ -transform.

Considering that an integral part of the Laplace transform on the Time Scale is the exponential function, in this paper we have given the definition and properties of this function. We have listed its forms on arbitrary Time Scales. In the following period, we will consider Laplace transform for each of the forms of the exponential function that we have set in this paper, i.e. which is valid for $\mathbb{T} = \mathbb{R}$; $\mathbb{T} = \ln\mathbb{N}$; $\mathbb{T} = \mathbb{N}_0^r, r \in (0, \frac{1}{2})$; $\mathbb{T} = \sqrt{\mathbb{N}_0}$; $\mathbb{T} = \mathbb{N}_0^r, r \in (\frac{1}{2}, \frac{2}{3})$; $\mathbb{T} = \mathbb{N}_0^{2/3}$; $\mathbb{T} = \mathbb{N}_0$; $\mathbb{T} = \mathbb{N}_0^r, r = 2, 3$; $\mathbb{T} = q^{\mathbb{N}_0}, q > 1$. Finally, we have given a concrete example of an exponential function over the delta derivative.

5. REFERENCES

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